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Extensions of some polynomial inequalities to the polar derivative ☆

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ABSTRACT

Let $p(z)$ be a polynomial of degree n and for any real or complex number α, let $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to α. In this paper, we obtain inequalities for the polar derivative of a polynomial having all its zeros inside or outside a circle. Our results shall generalize and sharpen some well-known polynomial inequalities.

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1. Introduction and statement of results

If $p(z)$ be a polynomial of degree n, then concerning the estimate of $|p'(z)|$ on the unit disk $|z| = 1$, we have

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The above inequality which is an immediate consequence of Bernstein's inequality on the derivative of a trigonometric polynomial is best possible with equality holding for the polynomial $p(z) = \lambda z^n$, λ being a complex number.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then the above inequality can be sharpened. In fact, Erdös conjectured and later Lax [13] proved that if $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

If the polynomial $p(z)$ of degree n has all its zeros in $|z| \leqslant 1$, then it was proved by Turán [17] that

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

The inequalities (1.2) and (1.3) are best possible and become equality for polynomials which have all its zeros on $|z| = 1$.

As an extension of (1.2) and (1.3) Malik [14] proved that if $p(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|, \tag{1.4}$$

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whereas if $p(z)$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.5}$$

In the literature, there already exist some refinements and generalizations of all the above inequalities, for example see Chan and Malik [7], Qazi [16], Gardner, Govil and Weems [9], Govil [10] etc.

By considering a more general class of polynomials $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \leqslant t \leqslant n$, not vanishing in $|z| < k$, $k \geqslant 1$, Gardner, Govil and Weems [9] proved

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+s_0} \Big\{ \max_{|z|=1} |p(z)| - m \Big\}, \tag{1.6}$$

where $m = \min_{|z|=k} |p(z)|$ and

$$s_0 = k^{t+1} \left\{ \frac{(\frac{t}{n}) \frac{|a_t|}{|a_0|-m} k^{t-1} + 1}{(\frac{t}{n}) \frac{|a_t|}{|a_0|-m} k^{t+1} + 1} \right\}. \tag{1.7}$$

The inequality (1.6) is of independent interest because, besides proving a generalization and refinements of (1.2), it also provides generalization and refinements of the results of Aziz and Dawood [2], Chan and Malik [7], Govil [10] and Malik [14].

On the other hand, for the class of polynomials $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$ of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, Aziz and Shah [4] proved

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |p(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |p(z)| \right\}. \tag{1.8}$$

For $\mu = 1$, inequality (1.8) reduces to an inequality due to Govil [10].

Let $D_\alpha p(z)$ denotes the polar derivative of the polynomial $p(z)$ of degree n with respect to the point α. Then

$$D_\alpha p(z) = np(z) + (\alpha - z) p'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left[\frac{D_\alpha p(z)}{\alpha} \right] = p'(z).$$

Here, we shall extend inequalities (1.6) and (1.8) to the polar derivative of a polynomial and thereby obtain generalizations of these results. Besides, we shall also prove a more general result which gives a result of Govil and McTume [11] as a special case. We first prove the following.

Theorem 1. *If $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \leqslant t \leqslant n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then for every real or complex number α with $|\alpha| \geqslant 1$*

$$\max_{|z|=1} |D_\alpha p(z)| \leqslant \frac{n}{1+s_0} \Big\{ \big(|\alpha| + s_0\big) \max_{|z|=1} |p(z)| - \big(|\alpha| - 1\big) m \Big\}, \tag{1.9}$$

where $m = \min_{|z|=k} |p(z)|$ and s_0 is as defined in (1.7).

Clearly, Theorem 1 generalizes inequality (1.6) and to obtain (1.6) from the above theorem, simply divide both sides of (1.9) by $|\alpha|$ and let $|\alpha| \to \infty$.

It is easy to verify, for example by the derivative test, that for every α with $|\alpha| \geqslant 1$, the function $(\frac{|\alpha|+x}{1+x}) \max_{|z|=1} |p(z)| - (\frac{|\alpha|-1}{1+x}) m$, is a non-increasing function of x. If we combine this fact with Lemma 2, according to which $s_0 \geqslant k^t$ for $t \geqslant 1$, we get the following corollary.

Corollary 1. *If $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \leqslant t \leqslant n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then for every real or complex number α with $|\alpha| \geqslant 1$*

$$\max_{|z|=1} |D_\alpha p(z)| \leqslant \frac{n}{1+k^t} \Big\{ \big(|\alpha| + k^t\big) \max_{|z|=1} |p(z)| - \big(|\alpha| - 1\big) m \Big\}, \tag{1.10}$$

where $m = \min_{|z|=k} |p(z)|$.

The above corollary is an extension and refinement of a result of Aziz [1] and for $t = 1$, it reduces to a result of Aziz and Shah [5].

Remark 1. Dividing both sides of inequality (1.10) by $|\alpha|$ and let $|\alpha| \to \infty$, we get a result of Aziz and Shah [6].

Next, as an application of Corollary 1, we prove the following generalization of (1.8).

Theorem 2. *If $p(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leqslant\mu\leqslant n$, is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\leqslant 1$, and δ is any real or complex number with $|\delta|\leqslant 1$, then for $|z|=1$*

$$\left|D_\delta p(z)\right|\leqslant n\left(\frac{k^\mu+|\delta|}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right|-n\left(\frac{1-|\delta|}{k^{n-\mu}(1+k^\mu)}\right)m, \tag{1.11}$$

where $m=\min_{|z|=k}|p(z)|$.

The result is best possible and equality in (1.11) holds for $p(z)=(z^\mu+k^\mu)^{n/\mu}$, where n is a multiple of μ and $\delta\geqslant 0$.

Remark 2. If we take $\delta=0$ in (1.11), we get for $|z|=1$

$$\left|np(z)-zp'(z)\right|\leqslant\frac{nk^\mu}{1+k^\mu}\max_{|z|=1}\left|p(z)\right|-\frac{nm}{k^{n-\mu}(1+k^\mu)}. \tag{1.12}$$

If $\max_{|z|=1}|p(z)|=|p(e^{i\phi})|$, then from (1.12), we get

$$\left|p'\left(e^{i\phi}\right)\right|\geqslant\frac{n}{1+k^\mu}\max_{|z|=1}\left|p(z)\right|+\frac{nm}{k^{n-\mu}(1+k^\mu)}. \tag{1.13}$$

Since $\max_{|z|=1}|p'(z)|\geqslant|p'(e^{i\phi})|$, then from (1.13), we immediately get inequality (1.8).

Remark 3. For $\mu=1$, Theorem 2 gives the corresponding generalization of a result due to Govil [10, Theorem 2] for $k\leqslant 1$.

Finally, we prove the following more general result.

Theorem 3. *Let $p(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leqslant\mu\leqslant n$, be a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\leqslant 1$. Then for every real or complex number α with $|\alpha|\geqslant k^\mu$, we have*

$$\begin{aligned}\max_{|z|=1}\left|D_\alpha p(z)\right|\geqslant{}&\frac{n(|\alpha|-k^\mu)}{(1+k^\mu)}\max_{|z|=1}\left|p(z)\right|+n\left(\frac{|\alpha|+1}{k^{n-\mu}(1+k^\mu)}\right)m\\&+n\left(\frac{k^\mu-A_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right|+\frac{n(A_\mu-k^\mu)}{k^n(1+k^\mu)}m,\end{aligned} \tag{1.14}$$

where $m=\min_{|z|=k}|p(z)|$ and

$$A_\mu=\frac{n(|a_n|-\frac{m}{k^n})k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n|-\frac{m}{k^n})k^{\mu-1}+\mu|a_{n-\mu}|}.$$

It is easy to see that Theorem 3 also provides a refinement of the following result due to Dewan, Singh and Lal [8].

Theorem A. *Let $p(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leqslant\mu\leqslant n$, be a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\leqslant 1$, then for every real or complex number α with $|\alpha|\geqslant k^\mu$, we have*

$$\max_{|z|=1}\left|D_\alpha p(z)\right|\geqslant\frac{n(|\alpha|-k^\mu)}{(1+k^\mu)}\max_{|z|=1}\left|p(z)\right|+\frac{n(|\alpha|+1)}{k^{n-\mu}(1+k^\mu)}\min_{|z|=k}\left|p(z)\right|. \tag{1.15}$$

In fact, excepting the case when $k=1$ or $\frac{\mu}{n}(\frac{|a_{n-\mu}|}{|a_n|-\frac{m}{k^n}})=k^\mu$, the bound obtained in Theorem 3 is always sharp than the bound obtained from Theorem A and for this it needs to show that

$$n\left(\frac{k^\mu-A_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right|+\frac{n(A_\mu-k^\mu)}{k^n(1+k^\mu)}m\geqslant 0,$$

which is equivalent to

$$n\left(\frac{k^\mu-A_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right|\geqslant\frac{n(k^\mu-A_\mu)}{k^n(1+k^\mu)}m. \tag{1.16}$$

In view of inequality (2.16) of Lemma 14, the above inequality become equivalent to

$$\max_{|z|=1}\left|p(z)\right|\geqslant\frac{m}{k^n}. \tag{1.17}$$

Now using (1.1) in Lemma 9, we get

$$\left|q'(z)\right|\leqslant k^tn\max_{|z|=1}\left|p(z)\right|-\frac{nk^t}{k^n}m=nk^t\left\{\max_{|z|=1}\left|p(z)\right|-\frac{m}{k^n}\right\},$$

which is true and hence inequality (1.17) holds.

2. Lemmas

We will need following lemmas to prove our theorems.

Lemma 1. *If $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then for $|z| = 1$*

$$\left|q'(z)\right| \geqslant s_0\left|p'(z)\right| + mn, \tag{2.1}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$, $m = \min_{|z|=k}|p(z)|$ and s_0 is as given in (1.7)*.*

The above lemma is due to Gardner, Govil and Weems [9].

Lemma 2. *If $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then*

$$s_0 \geqslant k^t, \tag{2.2}$$

where s_0 is given in (1.7)*.*

The above lemma is again due to Gardner, Govil and Weems [9].
The following lemma is due to Aziz and Rather [3].

Lemma 3. *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$ and $q(z) = z^n\overline{p(1/\bar{z})}$, then on $|z| = 1$*

$$\left|q'(z)\right| \leqslant S_\mu\left|p'(z)\right|, \tag{2.3}$$

where

$$S_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|},$$

and

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \leqslant k^\mu. \tag{2.4}$$

Lemma 4. *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then*

$$\max_{|z|=1}\left|p'(z)\right| \geqslant \frac{n}{1+k^\mu}\max_{|z|=1}\left|p(z)\right|. \tag{2.5}$$

The above lemma is due to Aziz and Shah [4].

Lemma 5. *Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for every real or complex number α with $|\alpha| \geqslant S_\mu$, we have*

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geqslant n\left(\frac{|\alpha| - S_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right|, \tag{2.6}$$

where S_μ is as given in Lemma 3*.*

Proof. If $q(z) = z^n\overline{p(1/\bar{z})}$, then $p(z) = z^n\overline{q(1/\bar{z})}$ and one can easily verify that for $|z| = 1$

$$\left|q'(z)\right| = \left|np(z) - zp'(z)\right|.$$

Now for every real or complex number α with $|\alpha| \geqslant S_\mu$

$$\left|D_\alpha p(z)\right| = \left|np(z) + (\alpha - z)p'(z)\right| \geqslant |\alpha|\left|p'(z)\right| - \left|np(z) - zp'(z)\right|,$$

which implies that for $|z| = 1$

$$\left|D_\alpha p(z)\right| \geqslant |\alpha|\left|p'(z)\right| - \left|q'(z)\right|. \tag{2.7}$$

Inequality (2.7) when combined with Lemma 3 gives

$$\left|D_\alpha p(z)\right| \geqslant \left(|\alpha| - S_\mu\right)\left|p'(z)\right|, \quad \text{for } |z| = 1. \tag{2.8}$$

The above inequality (2.8) in conjunction with Lemma 4 gives

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geqslant n\left(\frac{|\alpha| - S_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right|,$$

which proves Lemma 5 completely. □

The following result is due to Aziz and Shah [5].

Lemma 6. *If $p(z)$ is a polynomial of degree n and β is any real or complex number, then*

$$\left|D_\beta q(z)\right| = \left|n\overline{\beta}p(z) + (1-\overline{\beta}z)p'(z)\right| \quad \textit{for } |z| = 1, \tag{2.9}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$.

Lemma 7. *(See Marden [15, p. 49].) If all the zeros of an nth degree polynomial $p(z)$ lie in $|z| \leqslant k$, then for $|\alpha| \geqslant k$, the polar derivative $D_\alpha p(z)$ of $p(z)$ at the point α also has all its zeros in $|z| \leqslant k$.*

Lemma 8. *If $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then on $|z| = 1$*

$$\left|q'(z)\right| \geqslant k^t\left|p'(z)\right| + n\min_{|z|=k}\left|p(z)\right|, \tag{2.10}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$.

The above lemma is due to Aziz and Shah [6].

Lemma 9. *If $p(z) = a_n z^n + \sum_{\nu=t}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant t \leqslant n$, be a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then on $|z| = 1$, we have*

$$\left|q'(z)\right| \leqslant k^t\left|p'(z)\right| - \frac{n}{k^{n-t}}\min_{|z|=k}\left|p(z)\right|, \tag{2.11}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$.

Proof. If $p(z)$ has all its zeros in $|z| \leqslant k$, where $0 < k \leqslant 1$, then $q(z)$ has all its zeros in $|z| \geqslant 1/k$, $1/k \geqslant 1$. Hence on applying Lemma 8 to polynomial $q(z) = \bar{a}_n + \sum_{\nu=t}^{n} \bar{a}_{n-\nu} z^\nu$, we get

$$\left|p'(z)\right| \geqslant \frac{1}{k^t}\left|q'(z)\right| + n\min_{|z|=1/k}\left|q(z)\right| = \frac{1}{k^t}\left|q'(z)\right| + \frac{n}{k^n}\min_{|z|=k}\left|p(z)\right|,$$

which implies

$$k^t\left|p'(z)\right| \geqslant \left|q'(z)\right| + \frac{n}{k^{n-t}}\min_{|z|=k}\left|p(z)\right|,$$

which is equivalent to (2.11). □

The following lemma is a special case of a result due to Govil and Rahman [12].

Lemma 10. *If $p(z)$ is a polynomial of degree n, then*

$$\left|p'(z)\right| + \left|q'(z)\right| \leqslant n\max_{|z|=1}\left|p(z)\right| \quad \textit{for } |z| = 1, \tag{2.12}$$

where $q(z) = z^n\overline{p(1/\bar{z})}$.

Lemma 11. *The function*

$$S_\mu(x) = \frac{nxk^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{nxk^{\mu-1} + \mu|a_{n-\mu}|}, \tag{2.13}$$

where $k \leqslant 1$ and $\mu \geqslant 1$, is a non-increasing function of x.

Proof. The proof follows by considering the first derivative test for $S_\mu(x)$. $\square$

Lemma 12. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n and $p(z)=0$ in $|z|\leqslant k$, $k>0$, then $|q(z)|\geqslant \frac{m}{k^n}$ for $|z|\leqslant 1/k$, and in particular*

$$|a_n|>\frac{m}{k^n}, \tag{2.14}$$

where $m=\min_{|z|=k}|p(z)|$.

Proof. If $p(z)=0$ in $|z|\leqslant k$, then $q(z)\neq 0$ for $|z|\leqslant 1/k$. We can assume without loss of generality that $q(z)$ has no zero on $|z|=1/k$, for otherwise the result holds trivially. Since $q(z)$, being a polynomial, is analytic for $|z|\leqslant 1/k$ and has no zeros in $|z|\leqslant 1/k$, by Minimum Modulus Principle

$$\left|q(z)\right|\geqslant \min_{|z|=1/k}\left|q(z)\right| \quad \text{for } |z|\leqslant 1/k,$$

which implies

$$\left|q(z)\right|\geqslant \frac{1}{k^n}\min_{|z|=k}\left|p(z)\right| \quad \text{for } |z|\leqslant 1/k,$$

which in particular implies

$$|a_n|=\left|q(0)\right|>\frac{m}{k^n},$$

which is inequality (2.14). $\square$

Lemma 13. *If $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leqslant\mu\leqslant n$, is a polynomial of degree n such that $p(z)\neq 0$ for $|z|<k$, $k\geqslant 1$ and $m=\min_{|z|=k}|p(z)|$, then*

$$\frac{|a_\mu|k^\mu}{|a_0|-m}\leqslant\frac{n}{\mu}. \tag{2.15}$$

The above result is due to Gardner, Govil and Weems [9].

Lemma 14. *If $p(z)=a_n z^n+\sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1\leqslant\mu\leqslant n$, is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\leqslant 1$, then*

$$A_\mu\leqslant k^\mu, \tag{2.16}$$

where A_μ is as defined in Theorem 3.

Proof. Since $p(z)=0$ in $|z|\leqslant k$, $k\leqslant 1$, then $q(z)\neq 0$ for $|z|\leqslant 1/k$, $1/k\geqslant 1$. Hence applying Lemma 13 to the polynomial

$$q(z)=\bar{a}_n+\sum_{\nu=t}^{n}\bar{a}_{n-\nu}z^\nu,$$

we get

$$\frac{\mu}{n}\left(\frac{|a_{n-\mu}|}{|a_n|-m'}\right)\left(\frac{1}{k}\right)^\mu\leqslant 1, \tag{2.17}$$

where $m'=\min_{|z|=1/k}|q(z)|=\frac{m}{k^n}$.

It is easy to see from (2.17) that

$$\mu|a_{n-\mu}|\leqslant n\left(|a_n|-\frac{m}{k^n}\right)k^\mu,$$

which implies

$$\left\{\mu|a_{n-\mu}|-n\left(|a_n|-\frac{m}{k^n}\right)k^\mu\right\}\leqslant 0,$$

which is equivalent to

$$\left(k^{\mu-1}-k^\mu\right)\left\{\mu|a_{n-\mu}|-n\left(|a_n|-\frac{m}{k^n}\right)k^\mu\right\}\leqslant 0,$$

that is

$$n\left(|a_n|-\frac{m}{k^n}\right)k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}\leqslant\left(\mu|a_{n-\mu}|+n\left(|a_n|-\frac{m}{k^n}\right)k^{\mu-1}\right)k^{\mu},$$

from which inequality (2.16) follows. □

3. Proofs of the theorems

Proof of Theorem 1. On combining Lemmas 1 and 10, we get for $|z|=1$

$$s_0|p'(z)|+mn+|p'(z)|\leqslant n\max_{|z|=1}|p(z)|,$$

which is equivalent to

$$|p'(z)|\leqslant\frac{n}{1+s_0}\left\{\max_{|z|=1}|p(z)|-m\right\}\quad\text{for } |z|=1. \tag{3.1}$$

Since $q(z)=z^n\overline{p(1/\bar{z})}$, it is easy to verify that for $|z|=1$

$$|q'(z)|=|np(z)-zp'(z)|. \tag{3.2}$$

Also for every real or complex number α with $|\alpha|\geqslant 1$, the polar derivative of $p(z)$ with respect to α is

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).$$

This implies for $|z|=1$

$$\begin{aligned}|D_\alpha p(z)|&\leqslant|np(z)-zp'(z)|+|\alpha||p'(z)|\\&=|q'(z)|+|\alpha||p'(z)|\\&=|q'(z)|+|p'(z)|-|p'(z)|+|\alpha||p'(z)|\\&\leqslant n\max_{|z|=1}|p(z)|+(|\alpha|-1)|p'(z)|.\end{aligned} \tag{3.3}$$

Inequality (3.3) in conjunction with inequality (3.1) gives for $|z|=1$

$$|D_\alpha p(z)|\leqslant n\max_{|z|=1}|p(z)|+(|\alpha|-1)\left\{\frac{n}{1+s_0}\left(\max_{|z|=1}|p(z)|-m\right)\right\},$$

from which we can obtain Theorem 1. □

Proof of Theorem 2. By hypothesis the polynomial $p(z)$ has all its zeros in $|z|\leqslant k$, $k\leqslant 1$, therefore the polynomial $q(z)=z^n\overline{p(1/\bar{z})}$ has no zeros in $|z|<1/k$, $1/k\geqslant 1$. Applying Corollary 1 to the polynomial $q(z)$, we get for $|\alpha|\geqslant 1$

$$|D_\alpha q(z)|\leqslant\frac{n}{1+1/k^t}\left\{\left(|\alpha|+\frac{1}{k^t}\right)\max_{|z|=1}|q(z)|-(|\alpha|-1)\min_{|z|=1/k}|q(z)|\right\}\quad\text{for } |z|=1.$$

This implies for $|z|=1$

$$|D_\alpha q(z)|\leqslant\frac{n(|\alpha|k^t+1)}{1+k^t}\max_{|z|=1}|p(z)|-\frac{nk^t(|\alpha|-1)}{1+k^t}\frac{1}{k^n}\min_{|z|=k}|p(z)|. \tag{3.4}$$

Now from Lemma 6, it easily follows that

$$|D_\alpha q(z)|=|\alpha||D_{1/\bar{\alpha}}p(z)|\quad\text{for } |\alpha|\geqslant 1 \text{ and } |z|=1. \tag{3.5}$$

Using (3.5) in (3.4), we get for $|\alpha|\geqslant 1$ and $|z|=1$

$$|\alpha||D_{1/\bar{\alpha}}p(z)|\leqslant\frac{n(|\alpha|k^t+1)}{1+k^t}\max_{|z|=1}|p(z)|-\frac{nk^t(|\alpha|-1)}{1+k^t}\frac{1}{k^n}\min_{|z|=k}|p(z)|. \tag{3.6}$$

Replacing $1/\bar{\alpha}$ by δ, so that $|\delta|\leqslant 1$, we obtain from (3.6)

$$|D_\delta p(z)|\leqslant\frac{n(|\delta|+k^t)}{1+k^t}\max_{|z|=1}|p(z)|-\frac{n(1-|\delta|)}{k^{n-t}(1+k^t)}\min_{|z|=k}|p(z)|,$$

for $|z|=1$ and $|\delta|\leqslant 1$. This completes the proof of Theorem 2. □

Proof of Theorem 3. Since by hypothesis, the polynomial

$$p(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu},\quad 1\leqslant\mu\leqslant n,$$

has all its zeros in $|z|\leqslant k$, $k\leqslant 1$. If $p(z)$ has a zero on $|z|=k$, then $m=0$ and the result follows from Lemma 5 in this case. Henceforth, we suppose that all the zeros of $p(z)$ lie in $|z|<k$, $k\leqslant 1$, so that $m>0$.

Now $m\leqslant|p(z)|$ for $|z|=k$, therefore if λ is any real or complex number such that $|\lambda|<1$, then

$$\left|\frac{m\lambda z^n}{k^n}\right|<\left|p(z)\right|\quad\text{for } |z|=k.$$

Since all the zeros of $p(z)$ lie in $|z|<k$, it follows by Rouche's theorem that all the zeros of $p(z)-\frac{m\lambda z^n}{k^n}$ also lie in $|z|<k$. Hence, by Lemma 7 for $|\alpha|\geqslant k^\mu$, the polynomial

$$D_\alpha\left[p(z)-\frac{m\lambda z^n}{k^n}\right]=D_\alpha p(z)-\frac{\lambda mn\alpha z^{n-1}}{k^n}\tag{3.7}$$

also has all its zeros in $|z|<k$, $k\leqslant 1$ and for every λ with $|\lambda|<1$. This implies

$$\left|D_\alpha p(z)\right|\geqslant\frac{mn|\alpha||z|^{n-1}}{k^n}\quad\text{for } |z|\geqslant k \text{ and } |\alpha|\geqslant k^\mu.\tag{3.8}$$

Because if (3.8) is not true, then there is a point $z=z_0$ with $|z_0|\geqslant k$, such that

$$\left|\left\{D_\alpha p(z)\right\}_{z=z_0}\right|\leqslant\left|\frac{mn\alpha z_0^{n-1}}{k^n}\right|.$$

We choose

$$\lambda=\frac{k^n\{D_\alpha p(z)\}_{z=z_0}}{mn\alpha z_0^{n-1}},$$

so that $|\lambda|<1$ and with this choice of λ, from (3.7), we have

$$D_\alpha\left\{p(z)-\frac{m\lambda z^n}{k^n}\right\}=0,$$

where $|z_0|\geqslant k$, which contradicts the fact that all the zeros of $D_\alpha\{p(z)-\frac{m\lambda z^n}{k^n}\}$ lie in $|z|<k$, $k\leqslant 1$.

Now the polynomial $p(z)-\frac{m\lambda z^n}{k^n}$ has all its zeros in $|z|\leqslant k$, $k\leqslant 1$, we can apply Lemma 5 to $p(z)-\frac{m\lambda z^n}{k^n}$ and obtain for $|\alpha|\geqslant k^\mu\geqslant S'_\mu$

$$\left|D_\alpha\left\{p(z)-\frac{m\lambda z^n}{k^n}\right\}\right|\geqslant n\left(\frac{|\alpha|-S'_\mu}{1+k^\mu}\right)\left|p(z)-\frac{m\lambda z^n}{k^n}\right|\quad\text{for } |z|=1,\tag{3.9}$$

where

$$S'_\mu=\frac{n|a_n-\frac{m\lambda}{k^n}|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n|a_n-\frac{m\lambda}{k^n}|k^{\mu-1}+\mu|a_{n-\mu}|}.\tag{3.10}$$

Since for every λ with $|\lambda|<1$, we have

$$\left|a_n-\frac{m\lambda}{k^n}\right|\geqslant|a_n|-\frac{m|\lambda|}{k^n}\geqslant|a_n|-\frac{m}{k^n}\tag{3.11}$$

and $|a_n|>\frac{m}{k^n}$ by Lemma 12. Now combining (3.10), (3.11) and Lemma 11 for every λ with $|\lambda|<1$, we get

$$S'_\mu=\frac{n|a_n-\frac{m\lambda}{k^n}|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n|a_n-\frac{m\lambda}{k^n}|k^{\mu-1}+\mu|a_{n-\mu}|}\leqslant\frac{n(|a_n|-\frac{m}{k^n})k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n|-\frac{m}{k^n})k^{\mu-1}+\mu|a_{n-\mu}|}=A_\mu\quad\text{(say)}.\tag{3.12}$$

Let z_0 be on $|z|=1$ such that $|p(z_0)|=\max_{|z|=1}|p(z)|$. Therefore, from (3.7), (3.9) and (3.12), we have

$$\begin{aligned}\left|\left\{D_\alpha p(z)\right\}_{z=z_0}-\frac{\lambda mn\alpha z_0^{n-1}}{k^n}\right|&\geqslant n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\left|p(z_0)-\frac{m\lambda z_0^n}{k^n}\right|\geqslant n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\left\{\left|p(z_0)\right|-\frac{m|\lambda||z_0|^n}{k^n}\right\}\\&=\frac{n(|\alpha|-A_\mu)}{1+k^\mu}\left|p(z_0)\right|-n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\frac{m|\lambda|}{k^n}.\end{aligned}\tag{3.13}$$

If in (3.13), we choose the argument of λ such that

$$\left|\left\{D_\alpha p(z)\right\}_{z=z_0} - \frac{\lambda mn\alpha z_0^{n-1}}{k^n}\right| = \left|\left\{D_\alpha p(z)\right\}_{z=z_0}\right| - \frac{mn|\alpha||\lambda||z_0|^{n-1}}{k^n},$$

which easily follows from (3.8), we obtain

$$\left|\left\{D_\alpha p(z)\right\}_{z=z_0}\right| - \frac{mn|\alpha||\lambda||z_0|^{n-1}}{k^n} \geqslant n\left(\frac{|\alpha| - A_\mu}{1+k^\mu}\right)\left|p(z_0)\right| - n\left(\frac{|\alpha| - A_\mu}{1+k^\mu}\right)\frac{m|\lambda|}{k^n}. \tag{3.14}$$

Since z_0 lies on $|z| = 1$ and $|p(z_0)| = \max_{|z|=1} |p(z)|$, the inequality (3.14) is equivalent to

$$\left|\left\{D_\alpha p(z)\right\}_{z=z_0}\right| \geqslant n\left(\frac{|\alpha| - A_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right| - n\left(\frac{|\alpha| - A_\mu}{1+k^\mu}\right)\frac{m|\lambda|}{k^n} + \frac{mn|\alpha||\lambda|}{k^n},$$

which implies that

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geqslant n\left(\frac{|\alpha| - A_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right| - n\left(\frac{|\alpha| - A_\mu}{1+k^\mu}\right)\frac{m|\lambda|}{k^n} + \frac{mn|\alpha||\lambda|}{k^n}. \tag{3.15}$$

Now, if in (3.15) we make $|\lambda| \to 1$, we get

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geqslant n\left(\frac{|\alpha| - A_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right| + \frac{mn}{k^n}\left(\frac{|\alpha|k^\mu + A_\mu}{1+k^\mu}\right),$$

which is equivalent to

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geqslant \frac{n(|\alpha| - k^\mu)}{(1+k^\mu)}\max_{|z|=1}\left|p(z)\right| + n\left(\frac{|\alpha|+1}{k^{n-\mu}(1+k^\mu)}\right)m + n\left(\frac{k^\mu - A_\mu}{1+k^\mu}\right)\max_{|z|=1}\left|p(z)\right| + \frac{n(A_\mu - k^\mu)}{k^n(1+k^\mu)}m,$$

which is the required result. $\square$

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